

# On a Class of Two-Dimensional Einstein Finsler Metrics of Vanishing S-Curvature

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## Abstract

An  $(\alpha, \beta)$ -metric is defined by a Riemannian metric  $\alpha$  and 1-form  $\beta$ . In this paper, we study a known class of two-dimensional  $(\alpha, \beta)$ -metrics of vanishing S-curvature. We determine the local structure of those metrics and show that those metrics are Einsteinian (equivalently, isotropic flag curvature) but generally are not Ricci-flat.

**Keywords:**  $(\alpha, \beta)$ -Metric, Einstein Metric, S-Curvature, Flag Curvature

**MR(2000) subject classification:** 53B40

## 1 Introduction

In Finsler geometry, Einstein metrics are defined in a natural way as that in Riemann geometry. An  $n$ -dimensional Finsler metric  $F$  is called an Einstein metric if its Ricc curvature  $Ric$  is isotropic,

$$Ric = (n - 1)\lambda F^2,$$

where  $\lambda = \lambda(x)$  is a scalar function.  $F$  is called of Ricci constant, if  $\lambda = \text{constant}$ . In particular,  $F$  is called Ricci-flat if  $\lambda = 0$ . It is well known that in dimension  $n \geq 3$ , every Einstein Riemann metric is of Ricci constant, and every 3-dimensional Einstein Riemann metric is of constant sectional curvature. We do not know whether it is still true for any Finsler metrics. It has been shown that many Finsler metrics have such a similar property as Riemann metrics, among which, two important cases are Randers metrics (cf. [1]) and square metrics (cf. [3]). An Einstein square metric in  $n \geq 2$  is always Ricci-flat (cf. [3] [6]), but it is not necessarily the case for a Randers metric (cf. [1] [2]).

The S-curvature is originally introduced for the volume comparison theorem ([8]), and it is a non-Riemannian quantity which plays an important role in Finsler geometry (cf. [4] [5] [8]–[12]). For a Finsler manifold, the flag curvature is an analogue of sectional curvature for a Riemann manifold. The flag curvature and the S-curvature are closely related. It is proved that, for a Finsler manifold  $(M, F)$  of scalar flag curvature, if  $F$  is of isotropic S-curvature  $\mathbf{S} = (n + 1)c(x)F$  for a scalar function  $c(x)$  on  $M$ , then the flag curvature must be in the following form

$$\mathbf{K} = \frac{3c_{x^m}y^m}{F} + \tau(x), \quad (1)$$

where  $\tau(x)$  is a scalar function on  $M$  ([5]). Clearly, if  $F$  is of constant S-curvature and of scalar flag curvature, then by (1),  $\mathbf{K} = \tau(x)$  is isotropic and  $\mathbf{K} = \text{constant}$  in  $n \geq 3$ .

An  $(\alpha, \beta)$ -metric is defined by a Riemannian metric  $\alpha = \sqrt{a_{ij}(x)y^i y^j}$  and a 1-form  $\beta = b_i(x)y^i \neq 0$  on a manifold  $M$ , which can be expressed in the following form:

$$F = \alpha\phi(s), \quad s = \beta/\alpha,$$

where  $\phi(s)$  is a function satisfying certain conditions such that  $F$  is positive definite on  $TM - 0$  (see [7]). Some recent studies show that many Einstein  $(\alpha, \beta)$ -metrics are Ricci-flat. In [6], it proves that an Einstein  $(\alpha, \beta)$ -metric with  $\phi(s)$  being a non-linear polynomial must be Ricci-flat. By this result, it is ever believed that any Einstein  $(\alpha, \beta)$ -metrics with  $\phi(s)$  being non-linear analytic must be Ricci-flat. But this is not true. In [13], the present author studies a class of  $(\alpha, \beta)$ -metric  $F$  with  $\phi(s) = (1 + s)^p$ , where  $p \neq 0$  is a real number, and shows that a two-dimensional Einstein square-root metric  $F = \sqrt{\alpha(\alpha + \beta)}$  ( $p = 1/2$ ) is generally not Ricci-flat (also see the following Theorem 1.1).

Generally, two-dimensional Finsler metrics have some different special curvature properties from higher dimensions (cf. [12] [14]–[16]). By definition, every 2-dimensional Finsler metric is of scalar flag curvature  $\mathbf{K} = \mathbf{K}(x, y)$ , but generally  $\mathbf{K}$  is not isotropic. A two-dimensional metric  $F$  is an Einstein metric if and only if  $F$  is of isotropic flag curvature. By (1), if  $F$  is of constant S-curvature, then  $\mathbf{K}$  is isotropic. Conversely, if a Randers metric  $F = \alpha + \beta$  is of isotropic flag curvature, then  $F$  is of constant S-curvature ([1]). In [10], we construct a family of two-dimensional Randers metrics which are of isotropic flag curvature  $\mathbf{K} = \mathbf{K}(x) \neq \text{constant}$ . In [13], we prove that a 2-dimensional square-root metric  $F = \sqrt{\alpha(\alpha + \beta)}$  is of isotropic flag curvature if and only if  $F$  is of vanishing S-curvature. In [12], we investigate again the known characterization ([4]) for  $(\alpha, \beta)$ -metrics of isotropic S-curvature, and obtain one more class of two-dimensional  $(\alpha, \beta)$ -metrics of vanishing S-curvature with  $\phi(s)$  defined by (2) below. In this paper, we will study certain curvature properties of such a class of  $(\alpha, \beta)$ -metrics in the following theorem.

**Theorem 1.1** *Let  $F = \alpha\phi(s)$ ,  $s = \beta/\alpha$ , be a two-dimensional  $(\alpha, \beta)$ -metric with  $\phi(s)$  being defined by*

$$\phi(s) = \{(1 + k_1 s^2)(1 + k_2 s^2)\}^{\frac{1}{4}} e^{\int_0^s \tau(s) ds}, \quad (2)$$

where

$$\tau(s) := \frac{\pm \sqrt{k_2 - k_1}}{2(1 + k_1 s^2)\sqrt{1 + k_2 s^2}},$$

and  $k_1$  and  $k_2$  are constants with  $k_2 > k_1$ . Suppose  $F$  is of isotropic S-curvature. Then locally we have

$$\alpha = \frac{\sqrt{B}}{(1 + k_1 B)^{\frac{3}{4}}(1 + k_2 B)^{\frac{1}{4}}} \sqrt{\frac{(y^1)^2 + (y^2)^2}{u^2 + v^2}}, \quad (3)$$

$$\beta = \frac{B}{(1 + k_1 B)^{\frac{3}{4}}(1 + k_2 B)^{\frac{1}{4}}} \frac{uy^1 + vy^2}{u^2 + v^2}, \quad (4)$$

where  $B = B(x)$ ,  $u = u(x)$ ,  $v = v(x)$  are some scalar functions satisfying

$$u_1 = v_2, \quad u_2 = -v_1, \quad uB_1 + vB_2 = 0, \quad (5)$$

where  $u_i := u_{x^i}$ ,  $v_i := v_{x^i}$  and  $B_i := B_{x^i}$ . Further, the S-curvature  $\mathbf{S} = 0$  and the flag curvature  $\mathbf{K}$  is isotropic given by

$$\mathbf{K} = -\frac{(u^2 + v^2)\sqrt{1 + k_2 B}}{4B^2} \left\{ 2\sqrt{1 + k_1 B}(B_{11} + B_{22}) - \frac{(u^2 + v^2)(2 + 3k_1 B)}{B\sqrt{1 + k_1 B}} \left(\frac{B_1}{v}\right)^2 \right\}, \quad (6)$$

where  $B_{ij} := B_{x^i x^j}$ .

Theorem 1.1 gives a class of two-dimensional Einstein Finsler metrics, but generally they are not Ricci-flat. We can easily find  $u, v, B$  satisfying (5), for example,  $u = -x^2$ ,  $v = x^1$  and  $B = (x^1)^2 + (x^2)^2$ .

If  $v = 0$  in (6), then  $B_1/v$  can be replaced by  $-B_2/u$  since (5). Further,  $F$  is positively definite if and only if  $1 + k_1 b^2 > 0$  (Lemma 3.1 below). If  $B(= \|\beta\|_\alpha^2) = \text{constant}$ , then  $\alpha$  is flat and  $\beta$  is parallel with respect to  $\alpha$  (Lemma 3.2 below).

In [12], we prove that  $F$  in Theorem 1.1 is of isotropic S-curvature if and only if  $F$  is of vanishing S-curvature. Take  $k_1 = -1, k_2 = 0$ , then  $F$  in Theorem 1.1 becomes  $F = \sqrt{\alpha(\alpha + \beta)}$ , which is called a square-root metric ([13]). In [13], when we study a class of  $(\alpha, \beta)$ -metrics of Einstein-reversibility, Theorem 1.1 has been actually proved for  $F = \sqrt{\alpha(\alpha + \beta)}$ , and we also show that the converse is also true for square-root metrics, namely, Einstein square-root metrics must be of vanishing S-curvature.

The converse of Theorem 1.1 might be true. We have a conjecture: *if the two-dimensional  $(\alpha, \beta)$ -metric  $F$  defined by (2) is Einsteinian, then  $F$  must be of vanishing S-curvature.* Some special cases can be verified, for example,  $k_1 = -1, k_2 = 0$  as shown above. We have also verified another more complicated special case:  $k_1 = 0, k_2 = 4$ . In general case, we only need to prove (9) below holds if  $F$  defined by (2) is an Einstein metric.

## 2 Preliminaries

For a Finsler metric  $F$ , the Riemann curvature  $R_y = R^i_k(y) \frac{\partial}{\partial x^i} \otimes dx^k$  is defined by

$$R^i_k := 2 \frac{\partial G^i}{\partial x^k} - y^j \frac{\partial^2 G^i}{\partial x^j \partial y^k} + 2 G^j \frac{\partial^2 G^i}{\partial y^j \partial y^k} - \frac{\partial G^i}{\partial y^j} \frac{\partial G^j}{\partial y^k},$$

where  $G^i$  are called the geodesic coefficients as follows

$$G^i := \frac{1}{4} g^{il} \{ [F^2]_{x^k y^l} y^k - [F^2]_{x^l} \}.$$

Then the Ricci curvature **Ric** is defined by  $\mathbf{Ric} := R^k_k$ . A Finsler metric is called of scalar flag curvature if there is a function  $\mathbf{K} = \mathbf{K}(x, y)$  such that

$$R^i_k = \mathbf{K} F^2 (\delta^i_k - F^{-2} y^i y_k), \quad y_k := (1/2 F^2)_{y^i y^k} y^i. \quad (7)$$

In two-dimensional case, we have

$$\mathbf{K} = \frac{\mathbf{Ric}}{F^2}. \quad (8)$$

Under the Hausdorff-Busemann volume form  $dV = \sigma_F(x) dx^1 \wedge \dots \wedge dx^n$ , where

$$\sigma_F(x) := \frac{\text{Vol}(B^n)}{\text{Vol}\{(y^i) \in R^n | F(y^i \frac{\partial}{\partial x^i}|_x) < 1\}},$$

the S-curvature is defined by

$$\mathbf{S} := \frac{\partial G^m}{\partial y^m} - y^m \frac{\partial}{\partial x^m} (\ln \sigma_F).$$

$\mathbf{S}$  is said to be isotropic if there is a scalar function  $c(x)$  on  $M$  such that

$$\mathbf{S} = (n+1)c(x)F.$$

If  $c(x)$  is a constant, then  $F$  is called of constant S-curvature.

For a Riemannian  $\alpha = \sqrt{a_{ij} y^i y^j}$  and a 1-form  $\beta = b_i y^i$ , let

$$r_{ij} := \frac{1}{2}(b_{i|j} + b_{j|i}), \quad s_{ij} := \frac{1}{2}(b_{i|j} - b_{j|i}), \quad r^i_j := a^{ik} r_{kj}, \quad s^i_j := a^{ik} s_{kj},$$

$$q_{ij} := r_{im}s_j^m, \quad t_{ij} := s_{im}s_j^m, \quad r_j := b^i r_{ij}, \quad s_j := b^i s_{ij},$$

$$q_j := b^i q_{ij}, \quad r_j := b^i r_{ij}, \quad t_j := b^i t_{ij},$$

where we define  $b^i := a^{ij}b_j$ ,  $(a^{ij})$  is the inverse of  $(a_{ij})$ , and  $\nabla\beta = b_{i|j}y^i dx^j$  denotes the covariant derivatives of  $\beta$  with respect to  $\alpha$ . Here are some of our conventions in the whole paper. For a general tensor  $T_{ij}$  as an example, we define  $T_{i0} := T_{ij}y^j$  and  $T_{00} := T_{ij}y^i y^j$ , etc. We use  $a_{ij}$  to raise or lower the indices of a tensor.

**Lemma 2.1** ([1]) *By Ricci identities we have*

$$s_{ij|k} = r_{ik|j} - r_{jk|i} - b^l \bar{R}_{kl ij},$$

$$s^k_{0|k} = r^k_{k|0} - r^k_{0|k} + b^l \bar{R}l_{00},$$

$$b^k s_{0|k} = r_k s^k_0 - t_0 + b^k b^l r_{kl|0} - b^k b^l r_{k0|l},$$

$$s^k_{|k} = r^k_{|k} - t^k_k - r^i_j r^j_i - b^i r^k_{k|i} - b^k b^i \bar{R}ic_{ik}.$$

where  $\bar{R}$  denotes the Riemann curvature tensor of  $\alpha$ .

**Lemma 2.2** ([12]) *Let  $F = \alpha\phi(s)$ ,  $s = \beta/\alpha$ , be a two-dimensional  $(\alpha, \beta)$ -metric on a manifold  $M$  with  $\phi(s)$  being defined by (2). Then  $F$  is of isotropic  $S$ -curvature  $\mathbf{S} = 3c(x)F$  if and only if*

$$r_{ij} = \frac{3k_1 + k_2 + 4k_1 k_2 b^2}{4 + (k_1 + 3k_2)b^2} (b_i s_j + b_j s_i). \quad (9)$$

In this case,  $\mathbf{S} = 0$

**Lemma 2.3** ([10]) *Let  $\alpha = \sqrt{a_{ij}y^i y^j}$  be an  $n(\geq 2)$ -dimensional Riemann metric which is locally conformally flat. Locally we express  $a_{ij} = e^{2\sigma(x)}\delta_{ij}$ . Then  $W_0 = W_i y^i$  is a conformal 1-form of  $\alpha$  satisfying*

$$W_{0|0} = -2c\alpha^2,$$

where  $c = c(x)$  is a scalar function and the covariant derivative is taken with respect to the Levi-Civita connection of  $\alpha$ , if and only if

$$\frac{\partial W^i}{\partial x^j} + \frac{\partial W^j}{\partial x^i} = 0 \quad (\forall i \neq j), \quad \frac{\partial W^i}{\partial x^i} = \frac{\partial W^j}{\partial x^j} \quad (\forall i, j), \quad (10)$$

where  $W^i := a^{ij}W_j$ . In this case,  $c$  is given by

$$c(x) = -\frac{1}{2}[\tau(x) + W^r \sigma_r], \quad (\tau := \frac{\partial W^1}{\partial x^1}, \quad \sigma_i := \sigma_{x^i}). \quad (11)$$

### 3 Proof of Theorem 1.1

We first give a lemma to show the positively definite condition of the  $(\alpha, \beta)$ -metric defined by (2).

**Lemma 3.1** *Let  $F = \alpha\phi(s)$ ,  $s = \beta/\alpha$ , be a two-dimensional  $(\alpha, \beta)$ -metric on a manifold  $M$  with  $\phi(s)$  being defined by (2). Then  $F$  is positively definite on  $TM - 0$  if and only if*

$$1 + k_1 b^2 > 0. \quad (12)$$

*Proof :* In [7], it is shown that an  $(\alpha, \beta)$ -metric  $F = \alpha\phi(\beta/\alpha)$  is positively definite if and only if for  $|s| \leq b$ ,

$$\phi(s) > 0, \quad \phi(s) - s\phi'(s) > 0, \quad \phi(s) - s\phi'(s) + (b^2 - s^2)\phi''(s) > 0. \quad (13)$$

Now let  $\phi(s)$  be given by (2) in (13). Clearly we have  $\phi(s) > 0$  for  $|s| \leq b$ . It is easy to see that

$$F \text{ is defined on } TM - 0 \iff 1 + k_1 s^2 > 0 \iff (12).$$

In the following, we assume (12). Then we only need to prove the second and the third inequalities hold.

The second inequality in (13) is equivalent to

$$(\sqrt{1 + k_2 s^2} \mp \sqrt{k_2 - k_1} s)(\sqrt{1 + k_2 s^2} \pm \frac{\sqrt{k_2 - k_1}}{2} s) > 0.$$

Clearly, the above inequality holds by (12).

The third inequality in (13) is equivalent to

$$f(s^2)\sqrt{1 + k_2 s^2} + g(s) > 0, \quad (|s| \leq b), \quad (14)$$

where

$$f(s^2) := [2k_1 k_2 (k_1 + k_2) b^2 + 3k_1^2 + 2k_2^2 - k_1 k_2] s^4 + [k_1 (9k_2 - k_1) b^2 + 3k_2 + 5k_1] s^2 + (3k_2 + k_1) b^2 + 4,$$

$$g(s) := -4a_1(1 + k_1 b^2)(1 + k_2 s^2)^2 s, \quad a_1 := \pm \frac{\sqrt{k_2 - k_1}}{2}.$$

To prove (14), we first show  $f(s^2) > 0$  for  $|s| \leq b$ . Put

$$f(t) = \tilde{A}t^2 + \tilde{B}t + \tilde{C}, \quad (0 \leq t \leq b^2),$$

where

$$\tilde{A} := 2k_1 k_2 (k_1 + k_2) b^2 + 3k_1^2 + 2k_2^2 - k_1 k_2, \quad \tilde{B} := k_1 (9k_2 - k_1) b^2 + 3k_2 + 5k_1, \quad \tilde{C} := (3k_2 + k_1) b^2 + 4.$$

Then by (12) we have

$$f(0) = (3k_2 + k_1) b^2 + 4 > 0, \quad f(b^2) = 2(1 + k_1 b^2)(1 + k_2 b^2)(2 + k_1 b^2 + k_2 b^2) > 0.$$

If  $\tilde{A} = 0$ , then the above shows  $f(s^2) > 0$  for  $|s| \leq b$ . If  $\tilde{A} \neq 0$ , then we get

$$f(-\frac{\tilde{B}}{2\tilde{A}}) = \frac{a_1^4(1 + k_1 b^2)[(24k_2 - k_1)b^2 + 23]}{\tilde{A}}.$$

The above shows  $\tilde{A} > 0$  since  $f(0) > 0$  or  $f(b^2) > 0$ . Thus  $f(-\frac{\tilde{B}}{2\tilde{A}}) > 0$ . Therefore,  $f(s^2) > 0$  for  $|s| \leq b$  in this case.

Now we prove (14). Without loss of generality, we assume  $a_1 > 0$  and  $0 \leq s \leq b$ . Then (14) is equivalent to  $(1 + k_2 s^2)f^2(s^2) - g^2(s) > 0$ , which is equivalent to

$$h(t) := \hat{A}t^2 + \hat{B}t + \hat{C} > 0, \quad (0 \leq t := s^2 \leq b^2),$$

where

$$\hat{A} := 4k_1 k_2^2 (3k_2 + k_1) b^4 + 4k_2 (3k_2^2 + 3k_1^2 + 2k_1 k_2) b^2 - 6k_1 k_2 + 9k_1^2 + 13k_2^2,$$

$$\hat{B} := 4k_1 k_2 (9k_2 - k_1) b^4 + 2(9k_2 - k_1)(3k_1 + k_2) b^2 + 12k_1 + 20k_2, \quad \hat{C} := (4 + k_1 b^2 + 3k_2 b^2)^2.$$

Likewise, we easily get  $h(0) > 0$ ,  $h(b^2) > 0$ . Then in a similar way to the proof of  $f(t) > 0$  for  $0 \leq t \leq b^2$  above, we obtain  $h(t) > 0$  for  $0 \leq t \leq b^2$ . Q.E.D.

**Lemma 3.2** *Under the condition of Lemma 2.2, if there is a neighborhood  $U$  such that  $b = \text{constant}$ , then  $\alpha$  is flat and  $\beta$  is parallel with respect to  $\alpha$  in  $U$ . In particular, if  $2(1 - k_1 k_2 b^4) + (k_2 - k_1)b^2 = 0$  in  $U$ , then  $\alpha$  is flat and  $\beta$  is parallel with respect to  $\alpha$  in  $U$ .*

*Proof :* If there is a neighborhood  $U$  such that  $2(1 - k_1 k_2 b^4) + (k_2 - k_1)b^2 = 0$  in  $U$ , then  $b = \text{constant}$  in  $U$ . Since  $b = \text{constant}$  is equivalent to  $r_i + s_i = 0$ , by (9), we get  $b = \text{constant}$  is equivalent to

$$(1 + k_1 b^2)(1 + k_2 b^2)s_i = 0.$$

So by the positive definiteness of  $F$  (Lemma 3.1), we have  $s_i = 0$ . Since  $n = 2$ , we have (27) below. So  $s_{ij} = 0$ , namely,  $\beta$  is closed. Then by (9),  $\beta$  is parallel with respect to  $\alpha$ , and thus  $\alpha$  is flat. Q.E.D.

By Lemma 3.2 and continuity, we only need to have a discussion at those points  $x \in M$  with  $2(1 - k_1 k_2 b^4) + (k_2 - k_1)b^2 \neq 0$ .

Now we determine the local structure in Theorem 1.1 given by (3)–(5). Since  $F$  in Theorem 1.1 is of isotropic S-curvature,  $\beta$  satisfies (9) by Lemma 2.2. Define a Riemannian metric  $\tilde{\alpha}$  and 1-form  $\tilde{\beta}$  by

$$\tilde{\alpha} := \alpha, \quad \tilde{\beta} := (1 + k_1 b^2)^{-\frac{3}{4}}(1 + k_2 b^2)^{-\frac{1}{4}}\beta. \quad (15)$$

Under the deformation (15), a direct computation shows that (9) is reduced to

$$\tilde{r}_{ij} = 0. \quad (16)$$

So  $\tilde{\beta}$  is a Killing form with respect to  $\alpha$ . Locally we can express  $\alpha$  as

$$\alpha := e^\sigma \sqrt{(y^1)^2 + (y^2)^2}, \quad (17)$$

where  $\sigma = \sigma(x)$  is a scalar function. Then by (10) in Lemma 2.3, we have

$$\tilde{\beta} = \tilde{b}_1 y^1 + \tilde{b}_2 y^2 = e^{2\sigma}(u y^1 + v y^2), \quad (18)$$

where  $u = u(x), v = v(x)$  are a pair of scalar functions such that

$$f(z) = u + iv, \quad z = x^1 + ix^2$$

is a complex analytic function, and further it follows from (11) in Lemma 2.3 and (16) that  $u, v$  and  $\sigma$  satisfy the following PDEs:

$$u_1 = v_2, \quad u_2 = -v_1, \quad u_1 + u\sigma_1 + v\sigma_2 = 0. \quad (19)$$

Actually  $\sigma$  can be determined in terms of the triple  $(B, u, v)$ , where  $B := b^2$ . Firstly by (15) and then by (17) and (18) we get

$$\|\tilde{\beta}\|_\alpha^2 = \frac{B}{(1 + k_1 B)^{\frac{3}{2}}\sqrt{1 + k_2 B}}, \quad \|\tilde{\beta}\|_\alpha^2 = e^{2\sigma}(u^2 + v^2). \quad (20)$$

Therefore, by (20) we get

$$e^{2\sigma} = \frac{B}{(u^2 + v^2)(1 + k_1 B)^{\frac{3}{2}}\sqrt{1 + k_2 B}}. \quad (21)$$

Now it follows from (15), (17), (18) and (21) that (3) and (4) hold. By (21), we have

$$\sigma = \ln \frac{\sqrt{B}}{\sqrt{u^2 + v^2}(1 + k_1 B)^{\frac{3}{4}}(1 + k_2 B)^{\frac{1}{4}}}. \quad (22)$$

Plugging (22) into  $u_1 + u\sigma_1 + v\sigma_2 = 0$  and using  $u_1 = v_2$ ,  $u_2 = -v_1$  in (19), we can write  $u_1 + u\sigma_1 + v\sigma_2 = 0$  equivalently as

$$[2(1 - k_1 k_2 b^4) + (k_2 - k_1)b^2](uB_1 + vB_2) = 0. \quad (23)$$

By Lemma 3.2 and continuity, (23) implies

$$uB_1 + vB_2 = 0.$$

Thus (5) holds.

Next we will use (8) to prove that the flag curvature of  $F$  in Theorem 1.1 is given by (6). The expression of the Ricci curvature **Ric** for an  $(\alpha, \beta)$ -metric is generally very long (see [6]). We will not write out the expression of the Ricci curvature of  $F$  in Theorem 1.1. To compute **Ric** of  $F$ , we need to show the following quantities:

$$\begin{aligned} & r_{00}, r_{00|0}, r_0, r_{0|0}, r_m^m, r, b^k r_{00|k}, b^k s_{0|k}, b^k q_{0k}, q_{00}, q_0, \\ & t_{00}, t_0, t_m^m, s_0^2, s_m s^m, s_{0|m}^m, s_{0|0}, Ric_\alpha. \end{aligned} \quad (24)$$

**Lemma 3.3** *In dimension  $n = 2$ , for a pair  $\alpha = \sqrt{a_{ij}y^i y^j}$  and  $\beta = b_i y^i$ , there is a  $\theta = \theta(x)$  such that*

$$s_0^2 = \theta(b^2 \alpha^2 - \beta^2), \quad t_{00} = -\theta \alpha^2. \quad (25)$$

Further, by (25), we easily get

$$s_m s^m = \theta b^2, \quad t_0 = -\theta \beta, \quad t_m^m = -2\theta. \quad (26)$$

*Proof :* Fix a point  $x \in M$  and take an orthonormal basis  $\{e_i\}$  at  $x$  such that

$$\alpha = \sqrt{(y^1)^2 + (y^2)^2}, \quad \beta = b y^1.$$

Then

$$s_0 = s_2 y^2, \quad b^2 \alpha^2 - \beta^2 = b^2 (y^2)^2.$$

So for some  $\theta = \theta(x)$ , the first formula in (25) holds. Similarly, since  $n = 2$ , it can be easily verified that

$$s_{ij} = \frac{b_i s_j - b_j s_i}{b^2}. \quad (27)$$

Therefore, by (27), we have

$$t_{00} = -\frac{s_m s^m \beta^2 + b^2 s_0^2}{b^4}. \quad (28)$$

Then plugging  $s_0^2$  in (25) and  $s_m s^m$  in (26) into (28) yields  $t_{00}$  in (25). Q.E.D.

By (9), we can compute the following quantities:

$$r_{00}, r_{00|0}, r_0, r_{0|0}, r_m^m, r, b^k r_{00|k}, b^k q_{0k}, q_{00}, q_0$$

To name a few, we have

$$r_m^m = 0, \quad r = 0, \quad r_0 = \frac{(3k_1 + k_2 + 4k_1 k_2 b^2)b^2}{4 + (k_1 + 3k_2)b^2} s_0,$$

$$\begin{aligned}
r_{0|0} &= \frac{32(1+k_1b^2)(1+k_2b^2)[k_1k_2(3k_2+k_1)b^4+8k_1k_2b^2+3k_1+k_2]}{[4+(k_1+3k_2)b^2]^3}s_0^2 \\
&\quad + \frac{b^2(3k_1+k_2+4k_1k_2b^2)}{4+(k_1+3k_2)b^2}s_{0|0}, \\
q_{00} &= \frac{4k_1k_2b^2+3k_1+k_2}{4+(k_1+3k_2)b^2}(\beta t_0+s_0^2), \quad q_0 = \frac{(4k_1k_2b^2+3k_1+k_2)b^2}{4+(k_1+3k_2)b^2}t_0.
\end{aligned}$$

Finally, assuming (9) holds, we will use the formulas in Lemma 2.1 to compute the following quantities

$$b^k s_{0|k}, \quad s_{0|m}^m, \quad s_{0|0}.$$

Since  $n = 2$ , we always have

$$Ric_\alpha = \lambda \alpha^2, \quad (29)$$

where  $\lambda = \lambda(x)$  is a scalar function. By (9), we can get  $r^m$  and then we have

$$\begin{aligned}
r_{|m}^m &= \frac{32(1+k_1b^2)(1+k_2b^2)[k_1k_2(3k_2+k_1)b^4+8k_1k_2b^2+3k_1+k_2]}{[4+(k_1+3k_2)b^2]^3}s_ms^m \\
&\quad + \frac{(3k_1+k_2+4k_1k_2b^2)b^2}{4+(k_1+3k_2)b^2}s_{|m}^m.
\end{aligned} \quad (30)$$

Plugging (9), (29) and (30) into the fourth formula in Lemma 2.1 and using (26), we obtain

$$\begin{aligned}
s_{|m}^m &= \frac{8(1+k_1b^2)(1+k_2b^2)[(3k_2^2-k_1^2+6k_1k_2)b^4+4(k_1+3k_2)b^2+8]}{[4+(k_1+3k_2)b^2]^2[2(1-k_1k_2b^4)+(k_2-k_1)b^2]}\theta \\
&\quad + \frac{b^2[4+(k_1+3k_2)b^2]}{2[2(1-k_1k_2b^4)+(k_2-k_1)b^2]}\lambda.
\end{aligned} \quad (31)$$

Plugging (9) into the third formula in Lemma 2.1 and using (26), we get

$$b^k s_{0|k} = \frac{2[2(1-k_1k_2b^4)+(k_2-k_1)b^2]}{4+(k_1+3k_2)b^2}\theta\beta \quad (32)$$

By (9), we can first get  $r_{0|m}^m$  expressed in terms of  $t_0$ ,  $s_{|m}^m$ ,  $s_ms^m$  and  $b^ms_{0|m}$ . Then plugging  $r_{m|0}^m (=0)$ ,  $r_{0|m}^m$ , (31) and (32) into the second formula in Lemma 2.1 and using (26), we obtain

$$s_{0|m}^m = \frac{\beta}{2(1-k_1k_2b^4)+(k_2-k_1)b^2} \left\{ \frac{2A}{[4+(k_1+3k_2)b^2]^2}\theta + \frac{1}{2}[4+(3k_2+k_1)b^2]\lambda \right\}, \quad (33)$$

where

$$\begin{aligned}
A : &= 2k_1k_2(k_1^2-18k_1k_2-15k_2^2)b^6 + (3k_1^3-135k_1k_2^2-57k_1^2k_2-3k_2^3)b^4 \\
&\quad - (156k_1k_2+18k_1^2+18k_2^2)b^2 - 16(3k_1+k_2).
\end{aligned}$$

**Lemma 3.4** *In dimension  $n = 2$ , for a pair  $\alpha = \sqrt{a_{ij}y^iy^j}$  and  $\beta = b_iy^i$ , there holds*

$$s_{0|0} = b^m(r_{m0|0} - r_{00|m}) - \lambda(b^2\alpha^2 - \beta^2) + q_{00} - t_{00}, \quad (34)$$

where  $\lambda$  is the sectional curvature of  $\alpha$ .



*Proof :* By definition, we easily get

$$s_{0|0} = b^m s_{m0|0} + q_{00} - t_{00}.$$

It follows from Lemma 2.1 that

$$s_{i0|0} = r_{i0|0} - r_{00|i} - b^m \bar{R}_{0mi0}.$$

Since  $n = 2$ , we have

$$\bar{R}_{jmik} = \lambda(a_{jk}a_{mi} - a_{ij}a_{mk}).$$

Therefore, we easily obtain (34).

Q.E.D.

Now by (9), we can first get  $r_{m0|0}$  and  $r_{00|m}$ . Then plugging them into (34) and using (26) and (32), we obtain

$$\begin{aligned} s_{0|0} = & \frac{-[4 + (k_1 + 3k_2)b^2]\lambda}{2[2(1 - k_1k_2b^4) + (k_2 - k_1)b^2]}(b^2\alpha^2 - \beta^2) + 2\theta \left\{ \frac{[2(1 - k_1k_2b^4) + (k_2 - k_1)b^2]}{4 + (k_1 + 3k_2)b^2} \alpha^2 \right. \\ & \left. - \frac{A_1}{[4 + (k_1 + 3k_2)b^2]^2[2(1 - k_1k_2b^4) + (k_2 - k_1)b^2]}(b^2\alpha^2 - \beta^2) \right\}, \end{aligned} \quad (35)$$

where

$$\begin{aligned} A_1 : = & 4k_1k_2(k_1 + 3k_2)b^6[2k_1k_2b^2 + 3(k_1 - k_2)] + (6k_1^3 - 6k_2^3 - 18k_1^2k_2 - 174k_1k_2^2)b^4 \\ & - (12k_1^2 + 28k_2^2 + 216k_1k_2)b^2 - 24(3k_1 + k_2). \end{aligned}$$

Now we have obtained the expressions of all those quantities in (24), as shown in (25), (26), (29), (32), (33), (35), etc. Plug all those quantities and (9) into (8) and then the flag curvature  $\mathbf{K}$  of  $F$  in Theorem 1.1 is given by

$$\mathbf{K} = \frac{2(1 + k_2b^2)}{2(1 - k_1k_2b^4) + (k_2 - k_1)b^2} \left\{ \lambda - \frac{8A_2}{b^2[4 + (k_1 + 3k_2)b^2]^2} s_m s^m \right\}, \quad (36)$$

where

$$A_2 := 2k_1^2k_2^2b^6 + 5k_1k_2(k_1 + k_2)b^4 + k_1(k_1 + 13k_2)b^2 + 2(2k_1 + k_2).$$

In the final step, we will show (36) can be written as (6). Put  $\alpha$  and  $\beta$  as

$$\alpha = e^\sigma \sqrt{(y^1)^2 + (y^2)^2}, \quad \beta = e^\sigma (\xi y^1 + \eta y^2), \quad (37)$$

where  $\sigma = \sigma(x)$ ,  $\xi = \xi(x)$ ,  $\eta = \eta(x)$  are scalar functions. In the following, we define

$$\sigma_i := \sigma_{x^i}, \quad \sigma_{ij} := \sigma_{x^i x^j}, \quad etc.$$

The sectional curvature  $\lambda$  of  $\alpha$  and the norm  $b = \|\beta\|_\alpha$  are given by

$$\lambda = -e^{-2\sigma}(\sigma_{11} + \sigma_{22}), \quad b^2 = \xi^2 + \eta^2. \quad (38)$$

Further, we have

$$s_m s^m = \frac{(\xi^2 + \eta^2)(\xi_2 + \xi\sigma_2 - \eta\sigma_1 - \eta_1)^2}{4e^{2\sigma}}. \quad (39)$$

Now comparing (37) with (3) and (4), we get

$$\xi = \frac{u\sqrt{B}}{\sqrt{u^2+v^2}}, \quad \eta = \frac{v\sqrt{B}}{\sqrt{u^2+v^2}}, \quad \sigma = \ln \frac{\sqrt{B}}{\sqrt{u^2+v^2}(1+k_1B)^{\frac{3}{4}}(1+k_2B)^{\frac{1}{4}}}. \quad (40)$$

Plugging (40) into (39) and using  $u_1 = v_2$ ,  $u_2 = -v_1$ , we have

$$s_m s^m = \frac{[4 + (k_1 + 3k_2)B]^2 (uB_2 - vB_1)^2}{64B\sqrt{1+k_1B}(1+k_2B)^{\frac{3}{2}}}. \quad (41)$$

By (5) we have

$$u_1 = v_2, \quad u_2 = -v_1, \quad u_{11} + u_{22} = 0, \quad v_{11} + v_{22} = 0, \quad B_2 = -\frac{u}{v}B_1. \quad (42)$$

Plugging  $\sigma$  in (22) or (40) into  $\lambda$  in (38) and using (42), we get

$$\begin{aligned} \lambda = & -\frac{u^2+v^2}{4B^2\sqrt{1+k_2B}} \left\{ [2(1-k_1k_2b^4) + (k_2-k_1)b^2] \sqrt{1+k_1B}(B_{11}+B_{22}) \right. \\ & \left. + \frac{(u^2+v^2)B_1^2}{Bv^2(1+k_2B)\sqrt{1+k_1B}} T \right\}, \end{aligned} \quad (43)$$

where

$$T := 2k_1k_2B^3(k_1k_2B + k_1 - k_2) + (k_1^2 - k_2^2 - 8k_1k_2)B^2 - 4(k_1 + k_2)B - 2$$

Now plugging (41) and (43) into (36) and using (42), we obtain (6).

Q.E.D.

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